# Dynamical construction of Kähler-Einstein metrics

Hajime TSUJI

November 24, 2006


**Abstract**

In this article, we give a new construction of a Kähler-Einstein metric on a smooth projective variety with ample canonical bundle. This result can be generalized to the construction of a singular Kähler-Einstein metric on a smooth projective variety of general type which gives an AZD of the canonical bundle.

As a consequence, for a proper projective morphism $f : X \longrightarrow S$ (with connected fibers) such that a general fiber is of general type and a positive integer $m$, we construct a canonical singular hermitian metric $h_{E,m}$ on $f_*\mathcal{O}_X(mK_{X/S})$ with semipositive curvature in the sense of Nakano.
MSC: 53C25(32G07 53C55 58E11)


## Contents



# 1 Introduction

Let $X$ be a smooth projective $n$-fold with ample canonical bundle defined over $\mathbb{C}$. Then by the celebrated solution of Calabi's conjecture ([A, Y1]), there exists

a unique Kähler-Einstein form $\omega_E$ such that

$$-\mathrm{Ric}_{\omega_E} = \omega_E,$$

where $\mathrm{Ric}_{\omega_E}$ denotes the Ricci form of the Kähler manifold $(X, \omega_E)$.

On the other hand for a complex manifolds with very ample $L^2$ canonical forms, there exists a standard Kähler form called the Bergman Kähler form.

Let us explain more precisely. Let $M$ be a complex manifold of dimension $n$ such that the space of $L^2$ canonical forms

$$H^0_{(2)}(M, \mathcal{O}_M(K_M)) := \{\eta \in H^0(M, \mathcal{O}_M(K_M)) \mid (\sqrt{-1})^{n^2} \int_M \eta \wedge \bar{\eta} < \infty\}$$

gives a very ample linear system. Then $M$ admits a Bergman kernel,

$$B(z, w) := \sum_i \sigma_i(z) \cdot \overline{\sigma_i(w)} \qquad (z, w \in M),$$

where $\{\sigma_i\}$ is a complete orthonormal basis of $H^0_{(2)}(M, \mathcal{O}_M(K_M))$ with respect to the inner product;

$$(\eta, \eta') := (\sqrt{-1})^{n^2} \int_M \eta \wedge \bar{\eta}'.$$

And

$$\omega_B(z) := \sqrt{-1}\partial\bar{\partial} \log B(z, z) \qquad (z \in M)$$

is a Kähler form which is called the Bergman Kähler form on $M$. The same construction applies for the case of the adjoint bundle of a (possibly singular) hermitian line bundle $(L, h)$ on $M$ (see Section 3).

Both Kähler-Einstein metrics and Bergman metrics are determined uniquely by the complex structures. In this sense these metrics are canonical. Hence it is natural to study the relation of these metrics.

Recently S.K. Donaldson found a new construction of Kähler-Einstein metrics or more generally Kähler metrics with constant scalar curvature. Actually he found a strong connection between the existence of Kähler metrics with constant scalar curvature and the asymptotic stability of Hilbert points of projective embeddings ([Do]). In particular this implies the connection between the existence of Kähler-Einstein metrics and the asymptotic stability of Hillbert points of projective embeddings ([Do]).

Let us explain (a part of) his results. Let $X$ be a smooth projective variety and let $L$ be an ample line bundle on $X$. Then for every sufficiently large positive integer $m$, the linear system $|mL|$ gives a projective embbedding

$$\Phi_m : X \longrightarrow \mathbb{P}^{N_m}$$

given by

$$\Phi_m(x) := [\sigma_0^{(m)} : \cdots : \sigma_{N_m}^{(m)}],$$

where $\{\sigma_0^{(m)}, \cdots, \sigma_{N_m}^{(m)}\}$ is a basis of $H^0(X, \mathcal{O}_X(mL))$. Hence $\Phi_m$ depends on the choice of the basis. Let $\omega_{FS}$ denote the Fubini-Study Kähler form on $\mathbb{P}^{N_m}$. If for some choice of $\{\sigma_0^{(m)}, \cdots, \sigma_{N_m}^{(m)}\}$ the equality

$$\int_X \frac{\sigma_i^{(m)} \cdot \bar{\sigma}_j^{(m)}}{\sum_{i=0}^{N_m} \mid \sigma_i^{(m)} \mid^2} (\Phi_m^* \omega_{FS})^n = \delta_{ij}$$

holds for every $0 \leqq i, j \leqq N_m$ (i.e.,$\{\sigma_0^{(m)}, \cdots, \sigma_{N_m}^{(m)}\}$ is orthonormal with respect to the $L^2$-inner product with respect to the hermitian metric $(\sum_{i=0}^{N_m} \mid \sigma_i^{(m)} \mid^2)^{-2}$ on $mL$ and the volume form $(\Phi_m^* \omega_{FS})^n$), the Kähler form

$$\omega_m := \frac{1}{m} \Phi_m^* \omega_{FS}$$

is called **balanced (or critical)**. The Hilbert point of $\Phi_m(X)$ is stable, if and only if there exists a choice of the basis $\{\sigma_0^{(m)}, \cdots, \sigma_{N_m}^{(m)}\}$ such that $\Phi_m$ is balanced ([Z]). Donaldson's theorem is stated as follows.

**Theorem 1.1** *([Do, p.482, Theorem 3]) Let $X$ be a smooth projective variety and let $L$ be an ample line bundle on $X$. Suppose that $Aut(X, L)$ is discrete. If $X$ admits a Kähler form $\omega$ cohomologous to $2\pi c_1(L)$ with constant scalar curvature, then for every sufficiently large $m$, $\Phi_m(X)$ is stable (this property is called that $(X, L)$ is asymptotically stable). And the limit of the balanced Kähler forms $\{\omega_m\}$ exists in $C^\infty$-topology and the limit is a Kähler form with constant scalar curvature.* $\square$

In short Theorem 1.1 gives a construction of a Kähler form with constant scalar curvature as the limit of s sequence of balanced Kähler forms. And Theorem 1.1 is closely related to the asymptotic expansion of Bergman kernels ([C, Ze]).

In this article, we shall give a new construction of Kähler-Einstein forms with negative Ricci curvature as a limit of Bergman Kähler forms. The purpose of this article is to relate Kähler-Einstein forms and Bergman Kähler forms in the case of projective manifolds with ample canonical bundle or more generally projective manifolds of general type.

Let $X$ be a smooth projective $n$-fold with ample canonical bundle. Let $m_0$ be a positive integer such that :

1. $\mid mK_X \mid$ is very ample for every $m \geqq m_0$,
2. For every pseudoeffective singular hermitian line bundle $(L, h_L)$(cf. Definition 2.3 below), $\mathcal{O}_X(m_0K_X + L) \otimes \mathcal{I}(h_L)$ is globally generated.

The existence of such $m_0$ follows from Nadel's vanishing theorem ([N, p.561]).

Let $h_{m_0}$ be a $C^\infty$ hermitian metric on $m_0K_X$ with strictly positive curvature. Suppose that we have constructed $K_m$ and the $C^\infty$ hermitian metric $h_m$ on $mK_X$. Then we define

$$K_{m+1} := K(X, (m+1)K_X, h_m)$$

and

$$h_{m+1} := 1/K_{m+1},$$

where $K(X,(m+1)K_X, h_m)$ denotes (the diagonal part of) the Bergman kernel of $(m+1)K_X$ with respect to $h_m$ constructed as follows.

Let $\{\sigma_0^{(m+1)}, \cdots, \sigma_{N_{m+1}}^{(m+1)}\}$ be the complete orthonormal basis of $H^0(X, \mathcal{O}_X((m+1)K_X))$ with respect to the inner product

$$(\sigma,\tau) := (\sqrt{-1})^{n_2}\int_X h_m \cdot \sigma \wedge \bar{\tau} \qquad (\sigma,\tau \in H^0(X,\mathcal{O}_X((m+1)K_X))).$$

Then for $x \in X$ we define

$$\begin{aligned} K_{m+1}(x) &= K(X,(m+1)K_X,h_m)(x) \\ &:= \sum_{i=0}^{N_{m+1}} \mid \sigma_i^{(m+1)} \mid^2 (x), \end{aligned}$$

where for a global section $\sigma$ of $(m+1)K_X$, $\mid \sigma \mid^2$ denotes the global section $\sigma \cdot \bar{\sigma}$ of $(K_X \otimes \overline{K_X})^{\otimes (m+1)}$. We note that by the choice of $m_0$, $\mid (m+1)K_X \mid$ is very ample. Hence $h_{m+1} := 1/K_{m+1}$ is a $C^\infty$ hermitian metric on $(m+1)K_X$. Inductively we construct the sequences $\{h_m\}_{m \geqq m_0}$ and $\{K_m\}_{m>m_0}$. This is the same construction originated by the author in [T3].

The following theorem is the main result in this article.

**Theorem 1.2** *Let $X$ be a smooth projective $n$-fold with ample canonical bundle. Let $m_0$ and $\{h_m\}_{m>m_0}$ be the sequence of hermitian metrics as above. Then*

$$h_\infty := \liminf_{m\to\infty} \sqrt[m]{(m!)^n \cdot h_m}$$

*is a $C^\infty$ hermitian metric on $K_X$ such that*

$$\omega_\infty := \sqrt{-1}\Theta_{h_\infty}$$

*is a Kähler form on $X$ with*

$$-\mathrm{Ric}_{\omega_\infty} = \omega_\infty.$$

□

**Remark 1.3** *The existence of the limit $h_\infty$ has already been proved in [T3]. For the case of smooth projective varieties of non-general type see [T3, T5].* □

The construction of Kähler-Einstein form in Theorem 1.2 is more straightforward than the one in Theorem 1.1. And Theorem 1.2 seems to imply that the sequence of Kähler forms

$$\{\frac{\sqrt{-1}}{m}\Theta_{h_m}\}_{m\geqq m_0}$$

induced by the projective morphisms $\Phi^{(m)} : X \longrightarrow \mathbb{P}^{N_m}(m > m_0)$ defined by

$$\Phi^{(m)}(x) = [\sigma_0^{(m)}(x) : \cdots : \sigma_{N_m}^{(m)}(x)] \qquad (x \in X)$$

is asymptotically nearly balanced.

We can generalize Theorem 1.2 to the case of the maximal Kähler-Einstein current (cf. Definition 5.2) on a smooth projective variety of general type whose canonical bundle is not necessarily ample (Theorem 5.6) without any essential change. And this immediately implies the uniqueness of the maximal Kähler-Einstein currents on smooth projective varieties of general type (Theorem 5.1).

This enables us to deduce the logarithmic plurisubharmonicity of Kähler-Einstein volume forms on a projective family (cf. Theorem 1.4) by using the recent result on variation of Bergman kernels ([B1, B2, B3, T5]). And we are able to study the degeneration of Kähler-Einstein currents on projective families of varieties of general type.

**Theorem 1.4** *Let $f : X \longrightarrow S$ be a proper projective morphism with connected fibers betweem smooth varieties. Let $S^\circ$ denote the maximal Zariski dense subset of $S$ such that $f$ is smooth over $X^\circ := f^{-1}(S^\circ)$. Suppose that a general fiber of $f$ is a smooth projective variety of general type. Let $\omega_{E/S}$ be the family of relative maximal Kähler-Einstein currents on $X^\circ$ (cf. Definition 5.2). Let $h^\circ_E$ be the singular hermitian metric on $K_{X/S}|X^\circ$ defined by*

$$h^\circ_E := (\omega^n_{E/S})^{-1},$$

*where $n$ denotes the relative dimension of $f : X \longrightarrow S$. Then we have the followings :*

1. *$h^\circ_E$ extends to a singular hermitian metric $h_E$ on $K_{X/S}$.*
2. *The curvature current $\Theta_{h_E}$ of $h_E$ is semipositive on $X$.*

□

Theorem 1.4 implies the following refinement of Kawamata's positivity theorem ([Ka, p.57, Theorem 1]) for the direct image of the relative pluricanonical bundle in the case that a general fiber is a variety of general type.

**Theorem 1.5** *Let $f : X \longrightarrow S$ be a proper projective morphism with connected fibers betweem smooth varieties. Let $S^\circ$ denote the maximal Zariski dense subset of $S$ such that $f$ is smooth over $X^\circ := f^{-1}(S^\circ)$. Suppose that a general fiber of $f$ is a smooth projective variety of general type. Let $h_E$ be the singular hermitian metric on $K_{X/S}$ as in Theorem 1.4.*

*$F_m := f_*\mathcal{O}_X(mK_{X/S})$ is locally free on $S^\circ$ and $F_m \mid S^\circ$ carries the continuous hermitian metric $h_{E,m}$ defined by*

$$h_{E,m}(\sigma, \tau) := (\sqrt{-1})^{n^2} \int_{X_s} h^{m-1}_E \cdot \sigma \wedge \bar{\tau} \qquad (\sigma, \tau \in H^0(X_s, \mathcal{O}_{X_s}(mK_{X_s}))),$$

*where $n$ denotes the relative dimension of $f : X \longrightarrow S$. Then we have the followings.*

1. *The curvature $\Theta_{h_{E,m}}$ of $h_{E,m}$ is semipositive in the sense of Nakano.*
2. *Let $x \in S - S^\circ$ be a point and let $\sigma$ be a local holomorphic section of $F_m$ on a neighbourhood $U$ of $x$. Then $\sqrt{-1}\bar{\partial}\partial \log h_{E,m}(\sigma, \sigma)$ extends to a closed positive current across $(S - S^\circ) \cap U$.*

$\square$

**Remark 1.6** *If $K_{X/S}$ is relatively ample, then $h_{E,m}$ is $C^\infty$ on $S^\circ$.* $\square$

Theorem 1.4 has several applications. For example it immediately gives canonical positive line bundles on the moduli space of canonically polarized varieties with only canonical singularities. Such applications will be discussed in the subsequent papers because of the length.

We should note that the convergence in Theorem 1.2 is much weaker than in Theorem 1.1. And Theorem 1.2 does not say anything about Kähler forms with constant scalar curvature at the moment.

# 2 Preliminaries

In this section, we shall review the basic terminologies used in this paper.

## 2.1 Singular hermitian metrics

In this subection $L$ will denote a holomorphic line bundle on a complex manifold $M$.

**Definition 2.1** *A* **singular hermitian metric** *$h$ on $L$ is given by*

$$h = e^{-\varphi}\cdot h_0,$$

*where $h_0$ is a $C^\infty$ hermitian metric on $L$ and $\varphi \in L^1_{loc}(M)$ is an arbitrary function on $M$. We call $\varphi$ a weight function of $h$.* $\square$

The curvature current $\Theta_h$ of the singular hermitian line bundle $(L,h)$ is defined by

$$\Theta_h := \Theta_{h_0} + \sqrt{-1}\partial\bar{\partial}\varphi,$$

where $\partial\bar{\partial}$ is taken in the sense of a current. The $L^2$-sheaf $\mathcal{L}^2(L,h)$ of the singular hermitian line bundle $(L,h)$ is defined by

$$\mathcal{L}^2(L,h)(U) := \{\sigma \in \Gamma(U,\mathcal{O}_M(L)) \mid h(\sigma,\sigma) \in L^1_{loc}(U)\},$$

where $U$ runs over the open subsets of $M$. In this case there exists an ideal sheaf $\mathcal{I}(h)$ such that

$$\mathcal{L}^2(L,h) = \mathcal{O}_M(L)\otimes\mathcal{I}(h)$$

holds. We call $\mathcal{I}(h)$ the **multiplier ideal sheaf** of $(L,h)$. If we write $h$ as

$$h = e^{-\varphi}\cdot h_0,$$

where $h_0$ is a $C^\infty$ hermitian metric on $L$ and $\varphi \in L^1_{loc}(M)$ is the weight function, we see that

$$\mathcal{I}(h) = \mathcal{L}^2(\mathcal{O}_M, e^{-\varphi})$$

holds. For $\varphi \in L^1_{loc}(M)$ we define the multiplier ideal sheaf of $\varphi$ by

$$\mathcal{I}(\varphi) := \mathcal{L}^2(\mathcal{O}_M, e^{-\varphi}).$$

**Example 2.2** *Let $\sigma \in \Gamma(X, \mathcal{O}_X(L))$ be the global section. Then*

$$h := \frac{1}{\mid \sigma \mid^2} = \frac{h_0}{h_0(\sigma, \sigma)}$$

*is a singular hermitian metric on $L$, where $h_0$ is an arbitrary $C^\infty$-hermitian metric on $L$ (the right hand side is obviously independent of $h_0$). The curvature $\Theta_h$ is given by*

$$\Theta_h = 2\pi\sqrt{-1}(\sigma)$$

*where $(\sigma)$ denotes the current of integration over the divisor of $\sigma$.* $\square$

**Definition 2.3** *$L$ is said to be* **pseudoeffective***, if there exists a singular hermitian metric $h$ on $L$ such that the curvature current $\Theta_h$ is a closed positive current. Also a singular hermitian line bundle $(L, h)$ is said to be* **pseudoeffective***, if the curvature current $\Theta_h$ is a closed positive current.* $\square$

## 2.2 Analytic Zariski decompositions

Let $L$ be a pseudoeffective line bundle on a compact complex manifold $X$. To analyze the ring :

$$R(X, L) = \oplus_{m=0}^{\infty} H^0(X, \mathcal{O}_X(mL))$$

it is sometimes useful to introduce the notion of analytic Zariski decompositions.

**Definition 2.4** *Let $M$ be a compact complex manifold and let $L$ be a holomorphic line bundle on $M$. A singular hermitian metric $h$ on $L$ is said to be an analytic Zariski decomposition, if the followings hold.*

1. *$\Theta_h$ is a closed positive current,*
2. *for every $m \geq 0$, the natural inclusion*

$$H^0(M, \mathcal{O}_M(mL) \otimes \mathcal{I}(h^m)) \to H^0(M, \mathcal{O}_M(mL))$$

*is an isomorphim.* $\square$

**Remark 2.5** *If an AZD exists on a line bundle $L$ on a smooth projective variety $M$, $L$ is pseudoeffective by the condition 1 above.* $\square$

It is known that for every pseudoeffective line bundle on a compact complex manifold, there exists an AZD on $L$ (cf. [T1, T2, D-P-S]).

The advantage of the AZD is that we can handle pseudoeffective line bundle $L$ on a compact complex manifold $X$ as a singular hermitian line bundle with semipositive curvature current as long as we consider the ring $R(X, L)$.

# 3 Variation of adjoint line bundles

In this section we shall review the results in [T5].

## 3.1 Theorems of Maitani-Yamaguchi and Berndtsson

In 2004, Maitani and Yamaguchi proved the following theorem.

**Theorem 3.1** *([M-Y]) Let $\Omega$ be a pseudoconvex domain in $\mathbb{C}_z \times \mathbb{C}_w$ with $C^1$ boundary. Let $\Omega_t := \Omega \cap (\mathbb{C}_z \times \{t\})$ and Let $K(z,t)$ be the Bergman kernel function of $\Omega_t$.*

*Then $\log K(z,t)$ is a plurisubharmonic function on $\Omega$.* $\square$

Recently generalizing Theorem 3.1, B. Berndtsson proved the following higher dimensional and twisted version of Theorem 3.1.

**Theorem 3.2** *([B1]) Let $D$ be a pseudoconvex domain in $\mathbb{C}^n_z \times \mathbb{C}^k_t$. And let $\phi$ be a plurisubharmonic function on $D$. For $t \in \Delta$, we set $D_t := \Omega \cap (\mathbb{C}^n \times \{t\})$ and $\phi_t := \phi \mid D_t$. Let $K(z,t)(t \in \mathbb{C}^k_t)$ be the Bergman kernel of the Hilbert space*

$$A^2(D_t, e^{-\phi_t}) := \{f \in \mathcal{O}(\Omega_t) \mid \int_{D_t} e^{-\phi_t} \mid f \mid^2 < +\infty\}.$$

*Then $\log K(z,t)$ is a plurisubharmonic function on $D$.* $\square$

As in mensioned in [B2], his proof also works for a pseudoconvex domain in a locally trivial family of manifolds which admits a Zariski dense Stein subdomain.

Also he proved the following theorem.

**Theorem 3.3** *([B2, Theorem 1.1]) Let us consider a domain $D = U \times \Omega$ and let $\phi$ be a plurisubharmonic function on $D$. For simplicity we assume that $\phi$ is smooth up to the boundary and strictly plurisubharmonic in $D$. Then for each $t \in U$, $\phi_t := \phi(\cdot, t)$ is plurisubharmonic on $\Omega$. Let $A^2_t$ be the Bergman space of holomorphic functions on $\Omega$ with norm*

$$\| f \|^2 = \| f \|^2_t := \int_\Omega e^{-\phi_t} \mid f \mid^2 .$$

*The spaces $A^2_t$ are all equal as vector spaces but have norms that vary with $t$. Then "infinite rank" vector bundle $E$ over $U$ with fiber $E_t = A^2_t$ is therefore trivial as a bundle but is equipped with a notrivial metric. Then $(E, \| \ \|_t)$ is strictly positive in the sense of Nakano.* $\square$

In Theorem 3.2 the assumption that $D$ is a pseudoconvex domain in the product space is rather strong. And in Theorem 3.3, Berndtsson also assumed that $D$ is a product.

## 3.2 Variation of hermitian adjoint bundles

Recently Berndtsson and I have independently generalized Theorems 3.2 and 3.3 to the case of projective families. The strategis of the proofs of [B3] and [T5] are completely different. Although [T5] depends heavily on [B1, B2], the result of [T5], is substantially stronger than that of [B3]. In fact in [B3], Berndtsson only consider the case of smooth fibrations, but in [T5], the method applies to the case of general projective fibrations which posseses singular fibers.

**Theorem 3.4** *([T5]) Let $f : X \longrightarrow S$ be a projective family of projective varieties over a complex manifold $S$. Let $S^\circ$ be the maximal nonempty Zariski open subset such that $f$ is smooth over $S^\circ$.*

*Let $(L, h)$ be a singular hermitian line bundle on $X$ such that $\Theta_h$ is semipositive on $X$.*
*Let $K_s := K(X_s, K_X + L\mid_{X_s}, h\mid_{X_s})$ be the Bergman kernel of $K_{X_s} + (L \mid X_s)$ with respect to $h \mid X_s$ for $s \in S^\circ$. Then the singular hermitian metric $h_B$ of $K_{X/S} + L \mid f^{-1}(S^\circ)$ defined by*

$$h_B \mid X_s := K_s^{-1} (s \in S^\circ)$$

*has semipositive curvature on $f^{-1}(S^\circ)$ and extends on $X$ as a singular hermitian metric on $K_{X/S} + L$ with semipositive curvature current.* □

**Theorem 3.5** *([T5]) Let $f : X \longrightarrow S$ be a projective family of over a complex manifold $S$ such that $X$ is smooth. Let $S^\circ$ be a nonempty Zariski open subset such that $f$ is smooth over $S^\circ$. Let $(L, h)$ be a hermitian line bundle on $X$ such that $\Theta_h$ is semipositive on $X$. We define the hermitian metric $h_E$ on $E := f_*\mathcal{O}_X(K_{X/S} + L) \mid S^\circ$ by*

$$h_E(\sigma, \tau) := (\sqrt{-1})^{n^2} \int_{X_s} h \cdot \sigma \wedge \bar{\tau} \qquad (\sigma, \tau \in H^0(X_s, \mathcal{O}_{X_s}(K_{X_s} + L\mid_{X_s}))),$$

*where $n$ denotes the relative dimension of $f : X \longrightarrow S$. Let $S_0$ be the maximal Zariski open subset of $S$ such that $E \mid S_0$ is locally free. Then $(E, h_E)|S_0$ is semipositive in the sense of Nakano. Moreover if $\Theta_h$ is strictly positive, then $(E, h_E)|S_0$ is strictly positive in the sense of Nakano.*

*Let $t \in S - (S^\circ \cap S_0)$ be a point and let $\sigma$ be a local nonvanishing holomorphic section of $E$ on a neighbourhood $U$ of $t$. Then $\sqrt{-1}\bar{\partial}\partial \log h_E(\sigma, \sigma)$ extends as a closed positive current across $(S - (S^\circ \cap S_0)) \cap U$.* □

*Proof of Theorems 3.4 and 3.5.*

Let $f : X \longrightarrow S$ be a projective family. Since the statement is local we may assume that $S$ is the unit open ball $B$ with center $O$ in $\mathbb{C}^m$. We may also and do assume that the family $f : X \longrightarrow B$ is a restriction of a projective family

$$\hat{f} : \hat{X} \longrightarrow B(O, 2)$$

over the open ball $B(O, 2)$ of radius 2 with center $O$. Let

$$F : \hat{X} \times B(O, 2) \longrightarrow B(O, 2) \times B(O, 2)$$

the fiber space defined by

$$F(x, t) = (f(x), t).$$

Let $\varepsilon$ be a positive number less than 1. We set

$$T(\varepsilon) = \{(s, t) \in B(O, 2) \times B(O, 2) \mid t \in B, s \in B(t, \varepsilon)\}$$

and

$$X(\varepsilon) := F^{-1}(T(\varepsilon)).$$

Let

$$f_\varepsilon : X(\varepsilon) \longrightarrow B$$

be the family defined by

$$f_\varepsilon(x,t) = t.$$

Since for $(x,t) \in X(\varepsilon)$, $x \in f^{-1}(B(t,\varepsilon))$ holds, we see that

$$X(\varepsilon)_t := f^{-1}(B(t,\varepsilon))$$

holds. Hence we may consider $X(\varepsilon)_t$ as a family

$$\pi_{\varepsilon,t} : X(\varepsilon,t) \longrightarrow B(t,\varepsilon).$$

We note that $T(\varepsilon)$ is a domain of holomorphy in $\mathbb{C}^{2m}$. Hence $X(\varepsilon)$ is a pseudoconvex domain in $X \times B(O,2)$. Since $X \times B(O,2)$ is a product manifold, the proof of Theorem 3.2 works without any essential change in this case (cf. [B1]). Hence if we define $K_\varepsilon$ by

$$K_\varepsilon \mid X(\varepsilon)_t := K(X(\varepsilon)_t, K_X + L \mid X(\varepsilon)_t, h_L \mid X(\varepsilon)_t),$$

then

$$\sqrt{-1}\partial\bar{\partial}\log K_\varepsilon \geqq 0 \tag{1}$$

holds on $X(\varepsilon)$. We note that

$$\lim_{\varepsilon\downarrow 0} \mathrm{vol}(B(t,\varepsilon))\cdot K(X(\varepsilon)_t, K_X + L \mid X(\varepsilon)_t, h_L \mid X(\varepsilon)_t) = K(X_t, K_{X_t} + L \mid X_t, h \mid X_t) \tag{2}$$

holds. In fact, if we consider the family

$$\pi_{\varepsilon,t} : X(\varepsilon) \longrightarrow B(t,\varepsilon)$$

as a family over the unit open ball $B$ in $\mathbb{C}^m$ with center $O$ by

$$t' \mapsto \varepsilon^{-1}(t' - t),$$

the limit as $\varepsilon \downarrow 0$ is nothing but the trivial family $X_t \times B$. We note that for a $L$-valued canonical form $\sigma$ on $f^{-1}(B(t,\varepsilon))$,

$$\int_{B(t,\varepsilon)} h_E(\sigma,\sigma)$$

is nothing but the $L^2$-norm of the $L$-valued canonical form $\sigma$ with respect $h_L$ over $f^{-1}(B(t,\varepsilon))$ by Fubini's theorem, where we abbrebiate the standard Lebesgue measure on $\mathbb{C}^m$. Then the desired equality follows from the $L^2$-extension theorm ([O-T, O]) and the extremal property of the Bergman kernels.

Combining (1) and (2), we complete the proof of Theorem 3.4.

The proof of Theorem 3.5, is quite similar. First we note that

$$f_\varepsilon : X(\varepsilon) \longrightarrow B$$

is everywhere smooth.

For the moment, we shall assume that $E$ is locally free on $B(O,2)$. Then there exists a global generator $\{\sigma_1, \cdots, \sigma_r\}$ of $E$ on $B(O,2)$, where $r = \operatorname{rank} E$.

Then we see that every $t \in B$, the fiber of the vector bundle $(f_\varepsilon)_*\mathcal{O}_{X(\varepsilon)}(K_{X(\varepsilon)/B}+L)\otimes\mathcal{I}(h_L)$ at $t$ is canonically isomorphic to $\mathbb{C}^r\times\mathcal{O}(B(t,\varepsilon))$ in terms of the frame $\{\sigma_1,\cdots,\sigma_r\}$. And moreover the space $\mathcal{O}(B(t,\varepsilon))$ is canonically isomorphic to $\mathcal{O}(B(O,\varepsilon))$ by the parallel translation. In this case

$$E_\varepsilon := (f_{\varepsilon,(2)})_*\mathcal{O}_{X(\varepsilon)}(K_{X(\varepsilon)/B}\otimes p_1^*L\otimes\mathcal{I}(p_1^*h))$$

is a vector bundle of infinite rank on $B$, where

$$p_1 : X(\varepsilon)\longrightarrow X$$

denotes the first projection

$$p_1(x,t) = x, \qquad (x,t)\in X(\varepsilon)$$

and $(f_{\varepsilon,(2)})_*\mathcal{O}_{X(\varepsilon)}(K_{X(\varepsilon)/B}\otimes p_1^*L\otimes\mathcal{I}(p_1^*h))$ denotes the direct image of $L^2$-holomorphic sections.

By the same proof as Theorem 3.3, we see that the curvature current of $h_{E,\varepsilon}$ is well defined everywhere on $B$ and is semipositive in the sense of Nakano. Letting $\varepsilon$ tend to 0, the curvature $\Theta_{h_{E_\varepsilon}}$ converges to the curvature $\Theta_{h_E}$ operating on $E_t\otimes\mathcal{O}_{B,t}$ in the obvious manner for every $t$ such that $f$ is smooth over $t$. Hence $\Theta_{h_E}$ is semipositive in the sense of Nakano.

Let $t\in S-(S^\circ\cap S_0)$ be a point and let $\sigma$ be a local nonvanishing holomorphic section of $E$ on a neighbourhood $U$ of $t$. Since the assertion is local, for the proof we may and do assume $\sigma$ is defined on $B(O,2)$. The existence of the extension of the current $\sqrt{-1}\bar{\partial}\partial\log h_E(\sigma,\sigma)$ can be verified as follows.

Let us begin the following lemma which follows from [B-T, p.27,Corollary 7.3].

**Lemma 3.6** *([B-T, Corollary 7.3]) Let $\{T_k\}_{k=1}^\infty$ is a sequence of closed positive $(1,1)$ current on the unit open disk $\Delta$ in $\mathbb{C}$. Let $T_k = \sqrt{-1}\partial\bar{\partial}\varphi_k$. Suppose that $\varphi_k$ is $L^\infty_{loc}$ on $\Delta$ and $\{\varphi_k\}$ converges to a plurisubharmonic function $\varphi$ on the punctured disk $\Delta^* = \Delta-\{O\}$. Then $\{T_k\}$ converges to a closed positive current on $\Delta$.* $\square$

Let us consider the current

$$T_k := \sqrt{-1}\bar{\partial}_s\partial_s\log\{\frac{1}{\mathrm{vol}(B(s,1/k))}\int_{B(s,,1/k)}h_E(\sigma,\sigma)\}$$

on $B$ for $k\geqq 1$, where the integration is taken with respect to the standard Lebesgue measure on $\mathbb{C}^m$. We note that since

$$\frac{1}{\mathrm{vol}(B(s,1/k))}\int_{B(s,1/k)}h_E(\sigma,\sigma)$$

is plurisuperharmonic

$$\{\frac{1}{\mathrm{vol}(B(s,1/k))}\int_{B(s,1/k)}h_E(\sigma,\sigma)\}_{k=1}^\infty$$

is monotone increasing. By Lemma 3.6, we see that $\sqrt{-1}\bar{\partial}\partial\log h_E(\sigma,\sigma)$ is canonically extended across $t\in B$ where $E$ is locally free (here $m=\dim B$ may be bigger than 1, but we may use slicing by curves to apply Lemma 3.6).

Next we shall consider the extension across the point where $E$ is not locally free. In this case we just need to slice $B$ by complex curves passing through the curve. Since every torsion free coherent sheaf over a curve is always locally free, we see that $\sqrt{-1}\bar{\partial}\partial \log h_E(\sigma, \sigma)$ is canonically extended across all the points on $B$. The extension of $h_B$ in Theorem 3.4 is similar.

This completes the proof of Theorems 3.4 and 3.5. $\square$

# 4 Proof of Theorem 1.2

Let $X$ be a smooth projective $n$-fold with ample canonical bundle. Let $m_0$ be a positive integer such that :

1. $\mid mK_X \mid$ is very ample for every $m \geqq m_0$,
2. For every pseudoeffective singular hermitian line bundle $(L, h_L)$, $\mathcal{O}_X(m_0K_X + L) \otimes \mathcal{I}(h_L)$ is globally generated.

Let $h_{m_0}$ be a $C^\infty$ hermitian metric on $m_0K_X$ with strictly positive curvature. Let $\{h_m\}_{m \geqq m_0}$ and $\{K_m\}_{m > m_0}$ be the sequences of hermitian metrics and Bergman kernels constructed as in Section 1, i.e., $\{h_m\}_{m \geqq m_0}$ and $\{K_m\}_{m > m_0}$ are defined inductively by

$$K_{m+1} = K(X, K_X + mK_X, h_m)$$

and

$$h_{m+1} = 1/K_{m+1}.$$

Let $\omega_E$ be the Kähler-Einstein form on $X$ such that

$$-\mathrm{Ric}_{\omega_E} = \omega_E.$$

Let $dV_E = (n!)^{-1}\omega_E^n$ be the volume form associated with $(X, \omega_E)$.

**Lemma 4.1**

$$\limsup_{m \to \infty} \sqrt[m]{(m!)^{-n}K_m} \geqq (2\pi)^{-n} dV_E$$

*holds on $X$.* $\square$

*Proof.* Let us consider the hermitian line bundle $(K_X, dV_E)$ on $X$. Let $p \in X$ be a point. Then by the Kähler-Einstein condition, there exists a holomorphic normal coordinate $(U, z_1, \cdots, z_n)$ such that

$$dV_E^{-1} = \{\prod_{i=1}^{n}(1- \mid z_i \mid^2) + O(\parallel z \parallel^3)\} \cdot \mid dz_1 \wedge \cdots \wedge dz_n \mid^{-2} \tag{3}$$

holds. Suppose that

$$C_{m-1} \cdot dV_E^{m-1} \leqq K_{m-1}$$

holds on $X$ for some positive constant $C_{m-1}$. We note that

$$K_m(x) = \sup\{\mid \sigma \mid^2 (x); \sigma \in H^0(X, \mathcal{O}_X(mK_X)), (\sqrt{-1})^{n^2}\int_X h_{m-1} \cdot \sigma \wedge \bar{\sigma} = 1\} \tag{4}$$

holds for every $x \in X$, by the extremal property of the Bergman kernel (This is well known. See for example, [Kr, p.46, Proposition 1.3.16]). We note that for the open unit disk $\Delta = \{t \in \mathbb{C} \mid \mid t \mid < 1\}$,

$$\int_{\Delta} (1- \mid t \mid^2)^m dt \wedge d\bar{t} = \frac{2\pi}{m+1} \tag{5}$$

holds. Then by Hörmander's $L^2$-estimate of $\bar{\partial}$-operator, we see that there exists a positive constant $\lambda_m$ such that

$$(\lambda_m \cdot (2\pi)^{-n} \cdot m^n) \cdot C_{m-1} \cdot dV_E^m \leqq K_m \tag{6}$$

with

$$\lambda_m \geqq 1 - \frac{C}{\sqrt{m}},$$

where $C$ is a positive constant independent of $m$.

In fact this can be verified as follows. Let $x \in X$ be a point on $X$ and let $(U, z_1, \cdots, z_n)$ be the normal coordinate as above. We may assume that $U$ is biholomorphic to the polydisk $\Delta^n(r)$ of radius $r$ with center $O$ in $\mathbb{C}^n$ for some $r$ via $(z_1, \cdots, z_n)$.

Taking $r$ sufficiently small we may assume that there exists a $C^\infty$ function $\rho$ on $X$ such that

1. $\rho$ is identically 1 on $\Delta^n(r/3)$.
2. $0 \leqq \rho \leqq 1$.
3. $\mathrm{Supp}\,\rho \subset\subset U$.
4. $\mid d\rho \mid < 3/r$, where $\mid \quad \mid$ denotes the pointwise norm with respect to $\omega_E$.

We note that by the equation (3), the mass of $\rho\cdot(dz_1\wedge\cdots\wedge dz_n)^{\otimes m}$ concentrates around the origin as $m$ tends to infinity. Hence by (5) we see that the $L^2$-norm

$$\parallel \rho \cdot (dz_1 \wedge \cdots \wedge dz_n)^{\otimes m} \parallel$$

of $\rho \cdot (dz_1 \wedge \cdots \wedge dz_n)^{\otimes m}$ with respect to $(dV_E)^{-\otimes m}$ and $\omega_E$ is asymptotically

$$\parallel \rho \cdot (dz_1 \wedge \cdots \wedge dz_n)^{\otimes m} \parallel^2 \sim (\frac{2\pi}{m})^n \tag{7}$$

as $m$ tends to infinity , where $\sim$ means that the ratio of the both sides converges to 1. We set

$$\phi := n\rho \log \sum_{i=1}^{n} \mid z_i \mid^2 .$$

We may and do assume that $m$ is sufficiently large so that

$$m \cdot \omega_E + \sqrt{-1}\partial\bar{\partial}\phi > 0$$

holds on $X$.

By (7), the $L^2$-norm

$$\parallel \bar{\partial}(\rho \cdot (dz_1 \wedge \cdots \wedge dz_n)^{\otimes m}) \parallel_\phi$$

of $\bar{\partial}(\rho\cdot(dz_1\wedge\cdots\wedge dz_n)^{\otimes m})$ with respect to $e^{-\phi}\cdot(dV_E)^{-\otimes m}$ and $\omega_E$ satisfies the inequality

$$\parallel \bar{\partial}(\rho\cdot(dz_1\wedge\cdots\wedge dz_n)^{\otimes m}) \parallel_{\phi}^2 \leqq C_0\cdot(\frac{3}{r})^{2n+2}(\frac{2\pi}{m})^n \tag{8}$$

for every $m$, where $C_0$ is a positive constant independent of $m$.

By Hörmander's $L^2$-estimate applied to the adjoint line bundle of the hermitian line bundle $((m-1)K_X, e^{-\phi}\cdot dV_E^{-(m-1)})$, we see that for every sufficiently large $m$, there exists a $C^\infty$ solution of the equation ;

$$\bar{\partial}u = \bar{\partial}(\rho\cdot(dz_1\wedge\cdots\wedge dz_n)^{\otimes m})$$

such that

$$u(x) = 0$$

and

$$\parallel u \parallel_{\phi}^2 \leqq \frac{2}{m}\parallel \bar{\partial}(\rho\cdot(dz_1\wedge\cdots\wedge dz_n)^{\otimes m}) \parallel_{\phi}^2$$

hold, where $\parallel\ \parallel_{\phi}$'s denote the $L^2$ norms with respect to $e^{-\phi}\cdot dV_E^{-(m-1)}$ and $\omega_E$ respectively. Then $\rho\cdot(dz_1\wedge\cdots\wedge dz_n)^{\otimes m} - u$ is a holomorphic section of $mK_X$ such that

$$(\rho\cdot(dz_1\wedge\cdots\wedge dz_n)^{\otimes m} - u)(x) = (dz_1\wedge\cdots\wedge dz_n)^m$$

and

$$\parallel \rho\cdot(dz_1\wedge\cdots\wedge dz_n)^{\otimes m} - u \parallel^2 \leqq (1 + C_0\cdot(\frac{3}{r})^{2n+2}\sqrt{\frac{2}{m}})(\frac{2\pi}{m})^n.$$

Hence by induction on $m$, using (4) and (6), we see that there exist positive constants $C$ and $C'$ such that for every $m > m_0$

$$K_m \geqq C'(\prod_{k=m_0}^{m}(1-\frac{C}{\sqrt{k}}))\cdot(m!)^n\cdot(2\pi)^{-mn}\cdot dV_E^m$$

holds on $X$. This implies that

$$\limsup_{m\to\infty}\sqrt[m]{(m!)^{-n}K_m} \geqq (2\pi)^{-n}dV_E$$

holds on $X$. $\square$

**Lemma 4.2**

$$\int_X \sqrt[m]{K_m} \leqq (\prod_{k=m_0}^{m}(N_k+1))^{\frac{1}{m}}\cdot(\int_X \sqrt[m_0]{K_{m_0}})^{\frac{m_0}{m}}$$

*holds, where* $N_k := \dim \mid kK_X \mid = \dim H^0(X,\mathcal{O}_X(kK_X)) - 1$. $\square$

*Proof.* By Hölder's ineqality we have

$$\begin{aligned}
\int_X \sqrt[m]{K_m} &= \int_X \frac{K_m^{\frac{1}{m}}}{K_{m-1}^{\frac{1}{m-1}}} \cdot K_{m-1}^{\frac{1}{m-1}} \\
&\leqq (\int_X \frac{K_m}{K_{m-1}^{\frac{m}{m-1}}} \cdot K_{m-1}^{\frac{1}{m-1}})^{\frac{1}{m}} \cdot (\int_X K_{m-1}^{\frac{1}{m-1}})^{\frac{m-1}{m}} \\
&= (\int_X \frac{K_m}{K_{m-1}})^{\frac{1}{m}} \cdot (\int_X K_{m-1}^{\frac{1}{m-1}})^{\frac{m-1}{m}} \\
&= (N_m + 1)^{\frac{1}{m}} \cdot (\int_X K_{m-1}^{\frac{1}{m-1}})^{\frac{m-1}{m}}
\end{aligned}$$

Then continuing this process, by using

$$\int_X K_{m-1}^{\frac{1}{m-1}} \leqq (N_{m-1} + 1)^{\frac{1}{m-1}} \cdot (\int_X K_{m-2}^{\frac{1}{m-2}})^{\frac{m-2}{m-1}},$$

we have that

$$\int_X (K_m)^{\frac{1}{m}} \leqq \{(N_m + 1) \cdot (N_{m-1} + 1)\}^{\frac{1}{m}} \cdot (\int_X (K_{m-2})^{\frac{1}{m-2}})^{\frac{m-2}{m}}$$

holds. Continueing this process we obtain the lemma. □

Using Lemma 4.2, we obtain the following lemma.

**Lemma 4.3**

$$\limsup_{m\to\infty} \frac{1}{(m!)^{\frac{n}{m}}} \int_X (K_m)^{\frac{1}{m}} \leqq \frac{K_X^n}{n!}$$

*holds.* □

*Proof.* By the Kodaira vanishing theorem,

$$H^q(X, \mathcal{O}_X(mK_X)) = 0$$

holds for every $m \geqq 2$ and $q \geqq 1$. Then by the Hirzebruch Riemann-Roch theorem, we have that

$$N_m + 1 = \frac{K_X^n}{n!} m^n + O(m^{n-1})$$

holds. Then by Lemma 4.2, we see that

$$\limsup_{m\to\infty} \frac{1}{(m!)^{\frac{n}{m}}} \int_X (K_m)^{\frac{1}{m}} \leqq \frac{K_X^n}{n!}$$

holds. □

Combining Lemmas 4.1 and 4.2, we have the equality,

$$\limsup_{m\to\infty} \frac{1}{(m!)^{\frac{n}{m}}} \sqrt[m]{K_m} = (2\pi)^{-n} dV_E,$$

since
$$\int_X dV_E = \frac{1}{n!}\int_X \omega_E^n = \frac{(2\pi)^n K_X^n}{n!}$$
hold by the Kähler-Einstein condition. This completes the proof of Theorem 1.2. □

# 5 Dynamical construction of Kähler-Einstein currents

In this section we shall generalize Theorem 1.2 to the case of general smooth projective varieties of general type.

## 5.1 Existence of Kähler-Einstein currents

In this subsection, we shall review the existence of a Kähler-Einstein current on a smooth projective variety of general type. Allowing singularities, there are infinitely many choice of Kähler-Einstein metrics. But we focus on the metrics with minimal singularities.

**Theorem 5.1** *Let $X$ be a smooth projectie $n$-fold of general type. Then there exists a closed positive current $\omega_E$ on $X$ such that*

1. *$\omega_E$ is $C^\infty$ on a nonempty Zariski open subset $U$ of $X$.*
2. *$\omega_E$ is a Kähler-Einstein metric on $U$ with*
$$\omega_E = -\mathrm{Ric}_{\omega_E}$$
*on $U$.*
3. *The singular hermitian metric $(\omega_E^n)^{-1}$ is an AZD of $K_X$.*

□

**Definition 5.2** *Let $X$ be a smooth projectie $n$-fold of general type and let $\omega_E$ be the closed positive current on $X$ as in Theorem 5.1. We call $\omega_E$ the* **maximal Kähler-Einstein current** *on $X$.* □

**Remark 5.3** *Later we will see that the maximal Kähler-Einstein current is unique.* □

*Proof of Theorem 5.1.* The proof is more or less parallel to that of [S, Theorem 5.6, p.430]. Let $m_0$ be a sufficiently large positive integer such that $\mid m!K_X \mid$ gives a birational embedding of $X$ for every $m \geqq m_0$. Let $\pi_m : X_m \longrightarrow X$ be the resolution of Bs $\mid m!K_X \mid$ such that for every $m > m_0$
$$\pi_m : X_m \longrightarrow X$$
factors through $\pi_{m-1} : X_{m-1} \longrightarrow X$. Let
$$\mu_m : X_m \longrightarrow X_{m-1}$$

be the natural morphism. Let

$$\pi_m^* \mid m!K_X \mid = \mid P_m \mid + F_m$$

be the decomposition of $\pi_m^* \mid m!K_X \mid$ into the free part $\mid P_m \mid$ and the fixed component $F_m$. Let $V$ be an analytic subset of $X$ defined by

$V := \{x \in X \mid \Phi_{|m!K_X|}$ is not an embedding on an neighbourhood of $x$ for some $m \geqq m_0\}$.

There exists an effective $\mathbb{Q}$-divisor $E_m$ on $X_m$ respectively such that

1. $P_m - E_m$ is ample on $X_m$.
2. $\operatorname{Supp} E_m$ is contained in $\pi_m^{-1}(V)$.
3. $((m+1)!)^{-1}(P_{m+1} - E_{m+1}) - \mu_{m+1}^*(m!)^{-1}(P_m - E_m))$ is effective.

hold for every $m \geqq m_0$. After taking such a sequence $\{E_m\}$, we replace $\{E_m\}$ by $\{2^{-m}E_m\}$. Then it has the same properties as above. And we shall denote $\{2^{-m}E_m\}$ again by $\{E_m\}$.

Then by [S, Theorem 5.6], there exists a closed positive current $\omega_m$ on $X_m$ such that

1. $-\mathrm{Ric}_{\omega_m} = \omega_m$ holds on $X_m - \operatorname{Supp} E_m$,
2. The absolutely continuous part of $\omega_m$ represents $2\pi(m!)^{-1}(P_m - E_m)$
3. $(\pi_m)_*\omega_m$ represents the class $2\pi c_1(K_X)$,

Let us consider $\{\omega_m^n\}$ as a sequence of volume forms on $X - V$. And we shall identify $(\pi_m)_*\omega_m$ and $\omega_m$ on $X - V$. Then by the maximum principle we see that

$$\omega_m^n \leqq \omega_{m+1}^n$$

holds on $X - V$, by using the Einstein condition.

Let

$$\mu : Y \longrightarrow X$$

be a modification such that $\mu^{-1}(V)$ is a divisor with normal crossings. Let $H$ be a sufficiently ample divisor on $Y$ such that

$$D := \mu^{-1}(V) + H$$

is a divisor with normal crossings and $K_Y + D$ is ample. By [Kob], there exists a complete Kähler-Einstein form $\omega_D$ on $Y - D$. Let us consider $\omega_D$ as a complete Kähler form on $X - \mu(D)$. On the other hand by Yau's Schwarz lemma ([Y2]), we see that

$$\omega_m^n \leqq \omega_D^n$$

holds on $X - \mu(D)$ for every $m$. Hence by moving $D$,

$$\lim_{m\to\infty} \omega_m^n$$

exists on $X - V$.

Now we shall consider the uniform $C^2$-estimate on every compact subset of $X - V$. Let $F$ be an effective $\mathbb{Q}$-divisor on $X$ such that

1. $K_X - F$ is ample.

2. $\pi_m^* F - E_m$ is effective and $\mathrm{Supp}\,(\pi_m^* F - E_m)$ contains $\mathrm{Supp}\, E_m$.

The existence of such a divisor $F$ follows from the proof of Kodaira's lemma. Let $H$ be a smooth very ample divisor on $X$. Considering the exact sequence

$$0 \to H^0(X, \mathcal{O}_X(\ell K_X - H) \otimes \mathcal{I}_V) \to H^0(X, \mathcal{O}_X(\ell K_X)) \to H^0(X, \mathcal{O}_X(\ell K_X) \otimes \mathcal{O}_X/\mathcal{I}_H \cdot \mathcal{I}_V)$$

for $\ell >> 1$, we may find an effective member $F' \in \mid H^0(X, \mathcal{O}_X(\ell K_X - H)) \mid$. Then $K_X - \ell^{-1}F'$ is ample. Then $F$ can be taken as $\ell^{-1}F'$. By this argument we see that $\cap_F \mathrm{Supp}\, F = V$ holds, where $F$ runs all such $F$'s. Let

$$F = \sum a_i F_i$$

be the irreducible decomposition of $F$. Let $\sigma_i$ be a global section of $\mathcal{O}_X(F_i)$ with divisor $F_i$ respectively. Let $\Omega$ be a $C^\infty$-volume form on $X$. Then since $K_X - F$ is ample, there exist hermitian metrics $\{h_i\}$ of $\{\mathcal{O}_X(F_i)\}$ respectively such that

$$\omega_F := -\mathrm{Ric}\,\Omega - \sqrt{-1}\sum_i a_i \Theta_{h_i}$$

is a Kähler form on $X$. We note that $\pi_m^* F - E_m$ is effective. Let $u_m$ be a $C^\infty$-function on $X - D$ such that

$$\omega_m = \omega_F + \sqrt{-1}\partial\bar{\partial} u_m$$

and

$$\log \frac{(\omega_F + \sqrt{-1}\partial\bar{\partial} u_m)^n}{\Omega \cdot \prod_i \parallel \sigma_i \parallel^{2a_i}} = u_m$$

hold. We note that $u_m$ is identically $+\infty$ on $F$ by the choice of $F$. Hence there exists a point $p_0 \in X - F$, where $u_m$ takes its minimum. Then

$$\sqrt{-1}\partial\bar{\partial} \log \frac{(\omega_F + \sqrt{-1}\partial\bar{\partial} u_m)^n}{\Omega \cdot \prod_i \parallel \sigma_i \parallel^{2a_i}}(p_0) \geqq 0$$

holds. Hence

$$\omega_m(p_0) - \omega_F(p_0) \geqq 0$$

holds. This implies that

$$u_m(p_0) \geqq \log \frac{\omega_F^n}{\Omega \cdot \prod_i \parallel \sigma_i \parallel^{2a_i}}(p_0)$$

holds. Hence we see that

$$u_m(x) \geqq \log \frac{\omega_F^n}{\Omega \cdot \prod_i \parallel \sigma_i \parallel^{2a_i}}(p_0)$$

holds for every $x \in X - V$.

Similarly if $u_m + \sum_i 2a_i \log \parallel \sigma_i \parallel$ takes its maximum at a point $p_0'$ on $X - V$, since

$$\log \frac{\omega_m^n}{\Omega} = u_m + \sum_i 2a_i \log \parallel \sigma_i \parallel$$

holds. We note that since $\pi_m^* F - E_m$ is effective,

$$u_m + \sum_i 2a_i \log \parallel \sigma_i \parallel = -\infty$$

holds on $\operatorname{Supp} F$. This means that such a point $p_0'$ certainly exists. At $p_0'$ we have that

$$\sqrt{-1}\partial\bar{\partial} \log \frac{\omega_m^n}{\Omega}(p_0') \leqq 0$$

holds. Hence noting $\omega_m$ is Kähler-Einstein, we see that

$$\omega_m(p_0') \leqq (-\mathrm{Ric}\,\Omega)(p_0')$$

holds and

$$u_m + \sum_i 2a_i \log \parallel \sigma_i \parallel \leqq \log \frac{(-\mathrm{Ric}\,\Omega)^n}{\Omega}(p_0')$$

holds on $X$.

By the above consideration we have the following lemma.

**Lemma 5.4** *There exists a positive constant $C_0$ independent of $m$ such that*

$$-C_0 \leqq u_m \leqq C_0 - \sum_i 2a_i \log \parallel \sigma_i \parallel$$

*hold on $X - V$.* $\square$

**Lemma 5.5** *([T0, p. 127, Lemma 2.2])) We set*

$$f := \log \frac{\omega_F^n}{\Omega}.$$

*Let $C$ be a positive number such that*

$$C + \inf_{i \neq \ell} R_{i\bar{i}\ell\bar{\ell}} > 1$$

*holds on $X$, where $R_{i\bar{i}\ell\bar{\ell}}$ denotes the bisectional curvature.*

*Then*

$$e^{Cu_m}\Delta_m(e^{-Cu_m}(n + \Delta_F\, u_m)) \geqq (n + \Delta_F\, u_m)$$
$$+\Delta_F(f + \sum_i 2a_i \log \parallel \sigma_i \parallel) + (n + n^2 \inf_{i\neq\ell} R_{i\bar{i}\ell\bar{\ell}})$$
$$+C \cdot n(n + \Delta_F\, u_m) + (n + \Delta_F\, u_m)^{\frac{n}{n-1}} \exp(-\frac{1}{n-1}u_m + f)$$

*holds, where $\Delta_F$ denotes the Laplacian with respect to $\omega_F$ (i.e., $\Delta_F = \mathrm{trace}_{\omega_F}\sqrt{-1}\partial\bar{\partial}$) and $\Delta_m$ denotes the Laplacian with respect to $\omega_m$.*

$\square$

Let $x_0$ be the point where $e^{-Cu_m}(n + \Delta u_m)$ takes its maximum. Then

$$0 \leqq n + \Delta_F\, u_m(x_0) \leqq C_2$$

holds.

$$0 \leqq n + \Delta_F\, u_m \leqq \exp(C(u_m - u_m(x_0)) \cdot C_2$$

By Lemma 5.4, there exists a positive constant $C_3$ such that

$$n + \Delta_F\, u_m \leqq C_3(\prod_i \parallel \sigma_i \parallel^{2a_i})^{-C}$$

holds on $X - V$.

Applying the general theory of fully nonlinear elliptic equations ([Tr]), moving $F$, we get a uniform higher order estimate of $u_m$ on every compact subset of $X - V$. Letting $m$ tend to infinity, we see that by the monotonicity of $\{\omega_m^n\}$

$$\omega_E := \lim_{m\to\infty} \omega_m$$

exists in $C^\infty$-topology on every compact subset of $X - V$. Then it is clear that $\omega_E$ is Kähler-Einstein. By the construction of $\{E_m\}$ and the monotonicity of $\{\omega_m^n\}$, we see that $(\omega_E^n)^{-1}$ is an AZD of $K_X$. This completes the proof of Theorem 5.1. □

## 5.2 A generalization of Theorem 1.2

Let $X$ be a smooth projective $n$-fold of general type whose canonical bundle is not necessarily ample.

In this case we may also define the dynamical system of Bergman kernels as in the case that $X$ has ample canonical bundle. In fact the construction of the dynamical system of Bergman kernel is parallel except the following differences.

1. The starting line bundle is not a multiple of $K_X$, but a sufficiently ample line bundle.
2. The hermitian metrics $\{h_m\}$ are singular.

Let us explain in detail. Let $A$ be a very ample line bundle on $X$ such that for every pseudoeffective singular hermitian line bundle $(L, h_L)$,
$\mathcal{O}_X(m_0K_X + L) \otimes \mathcal{I}(h_L)$ is globally generated.

The existence of such $A$ follows from Nadel's vanishing theorem ([N, p.561]).

Let $h_0$ be a $C^\infty$-hermitian metric on $A$ with strictly positive curvature. Let $\{\sigma_0^{(1)}, \cdots, \sigma_{N_1}^{(1)}\}$ be a complete orthonormal basis of $H^0(X, \mathcal{O}_X(A + K_X))$ with respect to the inner product

$$(\sigma, \tau) := (\sqrt{-1})^{n_2} \int_X h_0 \cdot \sigma \wedge \bar{\tau} \qquad (\sigma, \tau \in H^0(X, \mathcal{O}_X(A + K_X))),$$

where we have considered $\sigma$ and $\tau$ as $A$-valued $(n, 0)$ forms. We set

$$K_1 = \sum_{i=0}^{N_1} |\sigma_i^{(1)}|^2$$

and

$$h_1 := 1/K_1.$$

It is clear that $K_1$ is independent of the choice of the orthonormal basis $\{\sigma_0^{(1)}, \cdots, \sigma_{N_1}^{(1)}\}$. Suppose that $h_m$ is defined for some $m \geqq m_0 + 1$. Then we define $h_{m+1}$ as follows. Let $\{\sigma_0^{(m+1)}, \cdots, \sigma_{N_{m+1}}^{(m+1)}\}$ be a complete orthonormal basis of $H^0(X, \mathcal{O}_X(A + (m+1)K_X))$ with respect to the inner product

$$(\sigma, \tau) := (\sqrt{-1})^{n_2} \int_X h_m \cdot \sigma \wedge \bar{\tau} \qquad (\sigma, \tau \in H^0(X, \mathcal{O}_X((m+1)K_X))).$$

Then we define

$$K_{m+1} = \sum_{i=0}^{N_{m+1}} |\sigma_i^{(m+1)}|^2$$

and

$$h_{m+1} := 1/K_{m+1}$$

inductively.

**Theorem 5.6** *Let $X$ be a smooth projective $n$-fold with ample canonical bundle. Let $A$ and $\{h_m\}_{m\geqq 1}$ be as above. Then*

$$h_\infty := \liminf_{m\to\infty} \sqrt[m]{(m!)^n \cdot h_m}$$

*is a hermitian metric on $K_X$ such that*

$$\omega_\infty := \sqrt{-1}\Theta_{h_\infty}$$

*is a Kähler-Einstein current $\omega_E$ on $X$ as in Theorem 5.1.* $\square$

The proof of Theorem 5.6 is essentially the same as the one of Theorem 1.2.
Similarly to Lemma 4.1, we obtain the following lower estimate.

**Lemma 5.7**

$$\limsup_{m\to\infty} \frac{1}{(m!)^{n/m}} \int_X (K_m)^{\frac{1}{m}} \geqq (2\pi)^{-n} dV_E$$

*holds.* $\square$

Lemma 5.7 can be obtained just as in the proof of Lemma 4.1. In the proof of Lemma 4.1, we have considered all the points on $X$, but here we only need to consider points on

$X^\circ := \{x \in X \mid \Phi_{|mK_X|}$ is an embedding on a neighbourhood of $x$ for some positive integer $m\}$.

$X^\circ$ is the locus where $\omega_E$ is $C^\infty$ strictly positive form. In fact the proof of Lemma 4.1 is essentially local (as $m$ tends to infinity).

For the upper estimate, we set

$$\mu(X, K_X) = n! \cdot \limsup_{m\to\infty} m^{-n} \dim H^0(X, \mathcal{O}_X(mK_X)).$$

Then by the same manner as the proof of Lemma 4.3, we obtain the following lemma.

**Lemma 5.8**

$$\limsup_{m\to\infty} \frac{1}{(m!)^{n/m}} (K_m)^{\frac{1}{m}} \leqq \frac{1}{n!}\mu(X, K_X).$$

*holds.* $\square$

Let $\pi_m : X_m \longrightarrow X$ be the resolution of $\mathrm{Bs} \mid m!K_X \mid$. Let

$$\pi_m^* \mid m!K_X \mid = \mid P_m \mid + F_m$$

be the decomposition of $\pi_m^* \mid m!K_X \mid$ into the free part $\mid P_m \mid$ and the fixed component $F_m$. By Fujita's theorem ([F, p.1, Theorem]), we see that

$$\lim_{m\to\infty} \frac{P_m^n}{(m!)^n} = \mu(X, K_X)$$

holds. Then by the construction of $\omega_E$ (cf. Section 5.1), we see that

$$\frac{1}{(2\pi)^n}\int_X dV_E = \frac{1}{n!}\mu(X, K_X)$$

holds. Hence combining Lemmas 5.7 and 5.8, we obtain that

$$\limsup_{m\to\infty} \frac{1}{(m!)^{n/m}}\int_X (K_m)^{\frac{1}{m}} = (2\pi)^{-n}\int_X dV_E$$

and

$$\limsup_{m\to\infty} \frac{1}{(m!)^{n/m}}(K_m)^{\frac{1}{m}} = (2\pi)^{-n} dV_E$$

hold. This completes the proof of Theorem 5.6. □

**Corollary 5.9** *Let $X$ be a smooth projective variety of general type. Then the maximal Kähler-Einstein current (cf. Definition 5.2) is unique.* □

# 6 Proof of Theorems 1.4 and 1.5

In this section we shall prove Theorems 1.4,1.5 by using Theorems 1.2,5.6. Roughly speaking, Theorems 1.2,5.6 imply that what we can say about Bergman kernels also holds for Kähler-Einstein volume forms.

*Proof of Theorems 1.4,1.5.* Let $A$ be a sufficiently ample line bundle on $X$ and let $h_0$ be a $C^\infty$ hermitian metric with strictly positive curvature.

Then for every $s \in S^\circ$, we define the dynamical system of the Bergman kernels $\{K_{m,s}\}$ on the fiber $X_s := f^{-1}(s)$ as in Section 5.2. Then we see that the hermitian metric

$$h_m \mid X_s = 1/K_{m,s}$$

on $A + mK_{X/S} \mid X^\circ$ has semipositive curvature by Theorem 3.4. And it extends to a singular hermitian metric on $A + mK_{X/S}$ with semipositive curvature as in Theorem 3.4. Then by Theorem 5.6, we see that $h_E$ is a singular hermitian metric on $K_{X/S}$ with semipositive curvature current. This completes the proof of Theorem 1.4. Then by Theorem 3.5, we complete the proof of Theorem 1.5. □

**Remark 6.1** *In [T5], I have proved the Nakano semipositivity of $f_*\mathcal{O}_X(mK_{X/S})$ similar to Theorem 1.5 even when a general fiber is of non-general type. But in this case the metric does not come from Kähler-Einstein metrics.* □

## References


[A] Aubin, T.: Equation du type Monge-Ampère sur les varieté kählerienne compactes, C.R. Acad. Paris **283** (1976), 459-464.

[B-T] Bedford, E. and Taylor, B.A., A new capacity of plurisubharmonic functions, Acta Math. **149** (1982), 1-40.

[B1] Berndtsson, B.: Subharmonicity properties of the Bergman kernel and some other functions associated to pseudoconvex domains, math.CV/0505469 (2005).

[B2] Berndtsson, B.: Curvature of vector bundles and subharmonicity of vector bundles, math.CV/050570 (2005).

[B3] Berndtsson, B.: Curvature of vector bundles associated to holomorphic fibrations, math.CV/0511225 (2005).

[C] Catlin, D.: The Bergman kernel and a theorem of Tian, Analysis and geometry in several complex variables (Katata 1997), 1-23, Trends in Math., Birkhäuser Boston, Boston MA. (1999).

[D-P-S] Demailly, J.P.-Peternell, T.-Schneider, M. : Pseudo-effective line bundles on compact Kähler manifolds, math. AG/0006025 (2000).

[Do] Donaldson, S.K.: Scalar curvarure and projective embeddings I, Journal of Differential Geom. **59** (2001), 479-522.

[F] Fujita, T.: Approximating Zariski deecomposition of big line bundle , Kodai Math. J. **17** (1994), 1-4.

[Ka] Kawamata, Y.: Kodaira dimension of Algebraic giber spaces over curves, Invent. Math. **66** (1982), pp. 57-71.

[Kob] Kobayashi, R.: Existence of Kähler-Einstein metrics on an open algebraic manifold, Osaka J. of Math. **21** (1984), 399-418.

[Kr] Krantz, S.: Function theory of several complex variables, John Wiley and Sons (1982).

[M-Y] Maitani, and Yamaguchi, S.: Variation of Bergman metrics on Riemann surfaces, Math. Ann. **330** (2004) 477-489.

[N] Nadel, A.M.: Multiplier ideal sheaves and existence of Kähler-Einstein metrics of positive scalar curvature, Ann. of Math. **132**(1990),549-596.

[O-T] Ohsawa, T and Takegoshi K.: $L^2$-extension of holomorphic functions, Math. Z. 195 (1987),197-204.

[O] Ohsawa, T.: On the extension of $L^2$ holomorphic functions V, effects of generalization, Nagoya Math. J. **161**(2001) 1-21.

[Ti] Tian, G.: On a set of polarized Kähler metrics on algebraic manifolds, Jour. Diff. Geom. **32**(1990),99-130.

[Tr] Trudinger, N.S.: Fully nonlinear elliptic equation under natural structure conditions, Trans. A.M.S. **272** (1983), 751-769.

[S] Sugiyama, K.: Einstein-Kahler metrics on minimal varieties of general type and an inequality between Chern numbers. Recent topics in differential and analytic geometry, 417–433, Adv. Stud. Pure Math., **18**-I, Academic Press, Boston, MA, 1990.

[T0] Tsuji H.: Existence and degeneration of Kahler-Einstein metrics on minimal algebraic varieties of general type. Math. Ann. **281** (1988), no. 1, 123–133.

[T1] Tsuji H.: Analytic Zariski decomposition, Proc. of Japan Acad. **61**(1992), 161-163.

[T2] Tsuji, H.: Existence and Applications of Analytic Zariski Decompositions, Trends in Math., Analysis and Geometry in Several Complex Variables(Katata 1997), Birkhäuser Boston, Boston MA.(1999), 253-272.

[T3] Tsuji, H.: Deformation invariance of plurigenera, Nagoya Math. J. **166** (2002), 117-134.

[T4] Tsuji, H.: Refined semipositivity and Moduli of canonical models, preprint (2005).

[T5] Tsuji, H: Variation of Bergman kernels of adjoint line bundles,math.CV/0511342 (2005).

[Y1] Yau, S.-T.: On the Ricci curvature of a compact Kähler manifold and the complex Mongé-Ampére equation, Comm. Pure Appl. Math. **31** (1978),339-441.

[Y2] Yau, S.-T.: A general Schwarz lemma for Kähler manifolds, Amer. J. of Math. **100** (1978), 197-203.

[Ze] Zelditch, S.: Szögo kernel and a theorem of Tian, International Reserch Notice 6 (1998), 317-331.

[Z] Zhang, S.: Heights and reductions of semistable varieties, Compositio Math. **104**(1996), 77-105.

Author's address
Hajime Tsuji
Department of Mathematics
Sophia University
7-1 Kioicho, Chiyoda-ku 102-8554
Japan
e-mail address: tsuji@mm.sophia.ac.jp